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Tijs, S.H.

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A CHARACTERIZATION OF THE VALUE OF LINEAR PROGRAMS

S.H. Tijs

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Abstract

We consider the value-function, defined on the set, consisting of those dual pairs of finite linear programs, for which at least one of the programs is feasible. It appears that this function is the only function, satisfying five properties: objectivity; symmetry; monotony; a property concerning deletion of rows; a property dealing with adjoining dominating columns.

1. In a nice paper [2], E.I. Vilkas considered the value-function on the set of finite matrix games, and characterized this function with the aid of four properties. The question arose, whether a similar result can be given for linear programs. An answer is given in sections 4 and 5 of this paper. In sections 2 and 3, the necessary notation is introduced.

2. Let A be an $m\times n$-matrix $(m,n \in \mathbb{N})$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Then

$P(A,b) := \{y \in \mathbb{R}^n;\ y \geq 0,\ Ay^t \leq b^t\}$ (*the feasible region of the primal program*),

$D(A,c) := \{x \in \mathbb{R}^m;\ x \geq 0,\ xA \geq c\}$ (*the feasible region of the dual program*),

$v_p(A,b,c) := \sup_{y\in P(A,b)} cy^t$ (*the value of the primal program*),

$v_d(A,b,c) := \inf_{x\in D(A,c)} xb^t$ (*the value of the dual program*).

It is well-known (see e.g. [1]) that

(2.1) $v_p(A,b,c) = v_d(A,b,c) \in \mathbb{R}$ iff $P(A,b) \neq \phi$ and $D(A,c) \neq \phi$.

(2.2) $v_p(A,b,c) = v_d(A,b,c) = \infty$ iff $P(A,b) \neq \phi$ and $D(A,c) = \phi$.

(2.3) $v_p(A,b,c) = v_d(A,b,c) = -\infty$ iff $P(A,b) = \phi$ and $D(A,c) \neq \phi$.

[As usual $\inf(\phi) := \infty$, $\sup(\phi) := -\infty$]

3. Let $m,n \in \mathbb{N}$ and let

$M(m,n)$ be the set of $m\times n$-matrices,

$L(m,n) := \{\langle A,b,c\rangle;\ A \in M(m,n),\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n\}$,

$F_p(m,n) := \{\langle A,b,c\rangle \in L(m,n);\ P(A,b) \neq \phi\}$,

$F_d(m,n) := \{\langle A,b,c\rangle \in L(m,n);\ D(A,c) \neq \phi\}$,

$F(m,n) := F_p(m,n) \cup F_d(m,n)$,

$F_p := \bigcup_{m,n\in\mathbb{N}} F_p(m,n)$, $F_d := \bigcup_{m,n\in\mathbb{N}} F_d(m,n)$ and $F := F_p \cup F_d$.

In view of (2.1), (2.2) and (2.3) we have

$$v_p(A,b,c) = v_d(A,b,c) \text{ for each } \langle A,b,c\rangle \in F.$$

This common value will also be denoted by $v(A,b,c)$.

4. In this section we mention some elementary properties of the value-function $v : F \to [-\infty,\infty]$. Most of them are well-known or easy to derive, but for convenience of the reader we also prove them.

(4.1)["*Objectivity*"]

Let $m \in \mathbb{N}$, $x \in \mathbb{R}^m$. Then $\langle [x^t],x,x_m\rangle \in F_p(m,1) \cap F_d(m,1)$ and $v([x^t],x,x_m) = x_m$.

[Here x^t is the transpose of the row vector x, $[x^t]$ is the $m\times 1$-matrix with column x^t and x_m is the m-th coordinate of x.]

Proof: $e_m := (0,0,\ldots,0,1) \in D([x^t],x_m)$, $1 \in P([x^t],x)$ and

$$v_d([x^t],x,x_m) \leq e_m x^t = x_m = x_m 1^t \leq v_p([x^t],x,x_m).$$

So, in view of (2.1), we have: $v([x^t],x,x_m) = x_m$. ||

(4.2) ["*Symmetry*"]

If $\langle A,b,c\rangle \in F_d$, then $\langle -A^t,-c,-b\rangle \in F_p$ and $v(A,b,c) = -v(-A^t,-c,-b)$.

[A^t is the transpose of A.]

Proof: Take $\langle A,b,c\rangle \in F_d$. Then $\phi \neq D(A,c) = P(-A^t,-c)$ and

$$v_d(A,b,c) = \inf_{x\in D(A,c)} xb^t = -\sup_{x\in P(-A^t,-c)} (-b)x^t = -v_p(-A^t,-c,-b). \ ||$$

Let $m,n \in \mathbb{N}$, $r \in \{1,\ldots,m+1\}$ and $k \in \{1,\ldots,n+1\}$. In the following we need the maps $\rho'_r : M(m+1,n) \to M(m,n)$, $\sigma'_k : M(m,n+1) \to M(m,n)$, $\sigma'_r : \mathbb{R}^{m+1} \to \mathbb{R}^m$, $\rho_r : L(m+1,n) \to L(m,n)$ and $\sigma_k : L(m,n+1) \to L(m,n)$ where

(a) $\rho'_r(A)$ is the $m\times n$-matrix which is obtained from the $(m+1)\times n$-matrix A by deleting the r-th row of A,

(b) $\sigma_k'(A)$ is the $m\times n$-matrix which we obtain from the $m\times(n+1)$-matrix A by deleting the k-th column of A,

(c) $\sigma_r'(x_1,\ldots,x_{m+1}) := (x_1,\ldots,x_{r-1},x_{r+1},\ldots,x_{m+1})$ for each $x \in \mathbb{R}^{m+1}$,

(d) $\rho_r(\langle A,b,c\rangle) := \langle\rho_r'(A),\sigma_r'(b),c\rangle$ for each $\langle A,b,c\rangle \in L(m+1,n)$,

(e) $\sigma_k(\langle A,b,c\rangle) := \langle\sigma_k'(A),b,\sigma_k'(c)\rangle$ for each $\langle A,b,c\rangle \in L(m,n+1)$.

(4.3) ["*Deleting a row does not decrease the value*"]

Let $m,n \in \mathbb{N}$ and let $\langle A,b,c\rangle \in F_p(m+1,n)$. Then for each $r \in \{1,\ldots,m+1\}$ we have

$$\rho_r(\langle A,b,c\rangle) \in F_p(m,n) \text{ and } v(A,b,c) \le v(\rho_r(\langle A,b,c\rangle)).$$

Proof: $\phi \ne P(A,b) \subset P(\rho_r'(A),\sigma_r'(b))$. So $\rho_r(\langle A,b,c\rangle) \in F_p(m,n)$ and

$$v_p(A,b,c) = \sup_{y\in P(A,c)} cy^t \le \sup_{y\in P(\rho_r'(A),\sigma_r'(b))} cy^t = v_p(\rho_r(\langle A,b,c\rangle)). \;\|$$

Definition

Let $m \in \mathbb{N}$, $n \in \mathbb{N}-\{1\}$, $\langle A,b,c\rangle \in L(m,n)$ and $k \in \{1,\ldots,n\}$. If there exist non-negative real numbers $u_1,\ldots,u_{k-1},u_{k+1},\ldots,u_n$ such that

$$Ae_k^t \ge \sum_{j\ne k} u_j Ae_j^t,\quad c_k \le \sum_{j\ne k} u_j c_j,$$

then we say that the k-th column of A *is dominating in* $\langle A,b,c\rangle$. [Ae_j^t is the j-th column of A.]

(4.4) ["*Adjoining a dominating column does not change the value*"]

Let $m,n \in \mathbb{N}$, $\langle A,b,c\rangle \in F_p(m,n)$ and $k \in \{1,\ldots,n+1\}$. Let $\langle C,b,q\rangle$ be an element of $L(m,n+1)$ such that

(4.4.1) $\sigma_k(\langle C,b,q\rangle) = \langle A,b,c\rangle$

(4.4.2) the k-th column of C is dominating in $\langle C,b,q\rangle$.

Then $\langle C,b,q\rangle \in F_p(m,n+1)$ and $v(C,b,q) = v(A,b,c)$.

Proof: (a) Take $y \in P(A,b)$. Then $\hat{y} := (y_1,\ldots,y_{k-1},0,y_k,\ldots,y_n) \in P(C,b)$ and $q\hat{y}^t = cy^t$ in view of (4.4.1). Hence $<C,b,q> \in F_p(m,n+1)$ and

$$v_p(C,b,q) \geq q\hat{y}^t = cy^t \text{ for each } y \in P(A,b)$$

and this implies that $v_p(C,b,q) \geq v_p(A,b,c)$.

(b) In view of (4.4.2) there are non-negative real numbers $u_1,\ldots,u_n$ such that

$$Ce_k^t \geq \sum_{j=1}^{n} u_j Ae_j^t,\ q_k \leq \sum_{j=1}^{n} u_j c_j.$$

For each $z = (z_1,\ldots,z_{k-1},\xi,z_k,\ldots,z_n) \in P(C,b)$ let

$$\overline{z} := (z_1+u_1\xi,\ldots,z_{k-1}+u_{k-1}\xi,z_k+u_k\xi,\ldots,z_n+u_n\xi) \geq 0.$$

Then

$$A\overline{z}^t = \sum_{j=1}^{n} z_j Ae_j^t + \xi \sum_{j=1}^{n} u_j Ae_j^t \leq \sum_{j=1}^{n} z_j Ae_j^t + \xi Ce_k^t = Cz^t \leq b.$$

So $\overline{z} \in P(A,b)$ and

$$qz^t = \sum_{j=1}^{n} c_j z_j + \xi q_k \leq \sum_{j=1}^{n} c_j(z_j+\xi u_j) = c\overline{z}^t \text{ for each } z \in P(C,b)$$

and this implies that $v_p(C,b,q) \leq v_p(A,b,c)$. ||

(4.5) ["*Monotony*"]

Let $m,n \in \mathbb{N}$, $<A,b,c> \in F_p(m,n)$, $<C,p,q> \in L(m,n)$ and suppose that

$$C \leq A,\ p \geq b \text{ and } q \geq c.$$

Then $<C,p,q> \in F_p(m,n)$ and $v(C,p,q) \geq v(A,b,c)$.

Proof: Take $y \in P(A,b)$. Then

$$Cy^t \leq Ay^t \leq b \leq p.$$

Hence $\phi \neq P(A,b) \subset P(C,p)$. Furthermore

$$v_p(C,p,q) \geq qy^t \geq cy^t \text{ for each } y \in P(A,b)$$

and this implies that $v_p(C,p,q) \geq v_p(A,b,c)$. ||

5. The following theorem shows that the properties (4.1)-(4.5) characterize the value-function $v : F \to [-\infty,\infty]$.

Theorem

Let $f : F \to [-\infty,\infty]$ be a function with the following five properties.

(5.1) For each $m \in \mathbb{N}$ and each $x \in \mathbb{R}^m$: $f([x^t],x,x_m) = x_m$.

(5.2) For each $<A,b,c> \in F_d$: $f(A,b,c) = -f(-A^t,-c,-b)$.

(5.3) For each $m,n \in \mathbb{N}$, $r \in \{1,\dots,m+1\}$ and each $<A,b,c> \in F_p(m+1,n)$:

$$f(A,b,c) \le f(\rho_r(<A,b,c>)).$$

(5.4) For each $m,n \in \mathbb{N}$, $k \in \{1,\dots,n+1\}$, $<A,b,c> \in F_p(m,n)$ and each $<C,b,q> \in F_p(m,n+1)$ such that the properties (4.4.1) and (4.4.2) hold we have: $f(C,b,q) = f(A,b,c)$.

(5.5) For each $m,n \in \mathbb{N}$, $<A,b,c> \in F_p(m,n)$ and $<C,p,q> \in F_p(m,n)$ with $C \le A$, $p \ge b$ and $q \ge c$ we have: $f(C,p,q) \ge f(A,b,c)$.

Then $f(A,b,c) = v(A,b,c)$ for each $<A,b,c> \in F$.

Proof: Take $m,n \in \mathbb{N}$ and $<A,b,c> \in F_p(m,n)$ and let α be a real number such that $\alpha < v_p(A,b,c)$. In this proof we need the matrices B, C and D and the vectors q, r and s where:

(a) $B := [A\ b^t] \in M_{m\times(n+1)}$. [B is obtained from A by adjoining the column b^t to A].

(b) C is the $m\times(n+1)$-matrix with $c_{ij} := b_{ij} \vee b_i$ for each $i \in \{1,\dots,m$ and each $j \in \{1,\dots,n+1\}$. [Here $b_{ij}(c_{ij})$ is the real number in the (i,j)-th entry of $B(C)$ and $b_{ij} \vee b_i$ is the maximum of the real numbers b_{ij} and b_i.]

(c) D is the $(m+1)\times(n+1)$-matrix which is obtained from C by adjoining to the $m\times(n+1)$-matrix C as $(m+1)$-th row the row $(\alpha,\alpha,\dots,\alpha) \in \mathbb{R}^{n\cdot}$

(d) $q := (c_1,c_2,\dots,c_n,\alpha) \in \mathbb{R}^{n+1}$.

(e) $r := (c_1 \wedge \alpha, c_2 \wedge \alpha,\dots,c_n \wedge \alpha,\alpha) \in \mathbb{R}^{n+1}$ [$c_i \wedge \alpha$ is the minimum of c_i and α].

(f) $s := (b_1,\ldots,b_m,\alpha) \in \mathbb{R}^{m+1}$.

We start by proving the following string of (in-)equalities:

(P.1) $\alpha = f([s^t],s,\alpha) = f(D,s,r) \leq f(C,b,r) \leq f(B,b,q) = f(A,b,c)$.

It follows from (5.1) that

(P.2) $\alpha = f([s^t],s,\alpha)$.

For each $k \in \{1,\ldots,n\}$ the k-th column De_k^t of D is also the k-th column of $\sigma'_{k+1} \circ \sigma'_{k+2} \circ \ldots \circ \sigma'_n(D)$ and

$$De_k^t \geq De_{n+1}^t,\ r_k = c_k \wedge \alpha \leq \alpha = r_{n+1}.$$

Hence De_k^t is dominating in $\sigma_{k+1} \circ \ldots \circ \sigma_n(\langle D,s,r\rangle)$.

Further $\langle [s^t],s,\alpha\rangle = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_n(\langle D,s,r\rangle)$.

Thus, by applying (4.4) and (5.4) n times, we may conclude

(P.3) $\langle D,s,r\rangle \in F_p(m+1,n+1)$ and $f(D,s,r) = f([s^t],s,\alpha)$.

Because $\langle C,b,r\rangle = \rho_{m+1}\langle D,s,r\rangle$, it follows from (4.3), (P.3) and (5.3) that

(P.4) $\langle C,b,r\rangle \in F_p(m,n+1)$ and $f(D,s,r) \leq f(C,b,r)$.

Note that $C \geq B$ and $r \leq q$. Hence, by (4.5) and (5.5), we have

(P.5) $\langle B,b,q\rangle \in F_p(m,n+1)$ and $f(B,b,q) \geq f(C,b,r)$.

Now take a $u \in P(A,b)$ such that $cu^t \geq \alpha$. Then

$$Be_{n+1}^t = b^t \geq Au^t = \sum_{j=1}^{n} u_j Ae_j^t = \sum_{j=1}^{n} u_j Be_j^t \text{ and}$$

$$q_{n+1} = \alpha \leq \sum_{j=1}^{n} u_j c_j = \sum_{j=1}^{n} u_j q_j$$

and this implies that the (n+1)-th column of B is dominating in $\langle B,b,q\rangle$. Then it follows from (5.4) that

(P.6) $f(A,b,c) = f(B,b,q)$.

Now (P.2)-(P.6) imply (P.1). We have proved that

$$f(A,b,c) \geq \alpha \text{ for each } \langle A,b,c\rangle \in F_p \text{ and each } \alpha < v_p(A,b,c).$$

Hence

(P.7) $\quad f(A,b,c) \geq v_p(A,b,c)$ for each $\langle A,b,c\rangle \in F_p$.

Now let $\langle A,b,c\rangle \in F_d$. Then $\langle -A^t,-c,-b\rangle \in F_p$; and by (5.2), (P.7) and (4.2) we have

$$f(A,b,c) = -f(-A^t,-c,-b) \leq -v_p(-A^t,-c,-b) = v_d(A,b,c).$$

So

(P.8) $\quad f(A,b,c) \leq v_d(A,b,c)$ for each $\langle A,b,c\rangle \in F_d$.

The properties (2.1), (P.7) and (P.8) imply that

(P.9) $\quad f(A,b,c) = v(A,b,c)$ for each $\langle A,b,c\rangle \in F_p \cap F_d$.

It follows from (2.2) and (P.7) that

(P.10) $\quad \infty = v(A,b,c) = f(A,b,c)$ for each $\langle A,b,c\rangle \in F_p - F_d$.

From (2.3) and (P.8) we may conclude

(P.11) $\quad -\infty = v(A,b,c) = f(A,b,c)$ for each $\langle A,b,c\rangle \in F_d - F_p$.

Since $F = (F_p \cap F_d) \cup (F_p - F_d) \cup (F_d - F_p)$ the conclusion of the theorem follows from (P.9), (P.10) and (P.11). ||

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Department of Mathematics
Catholic University
Toenooiveld
6525 ED Nijmegen
The Netherlands

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